§ 3 Hilbert's Axioms

Goal: Present a set of axioms for geometry proposed by Hilbert in 1899 which satisfies modern standard of rigor to supply the foundation for Euclid's geometry.

◂ David Hilbert (1862-1943)

German Mathematican.

Language in sets:

Space : set S

Points : elements of S

Lines : a subset of S

$\mathcal{L}$ = set of all lines

$(S, \mathcal{L})$: a geometry (or geometric model)

Instead of giving definitions of points and lines, we leave them as "undefined objects" and regard them as elements or subsets of a set S. Then, we require them to obey certain axioms.

3.1 Axioms of Incidence

Definition 3.1.1

A point P is said to be lying on a line ℓ if $P \in \ell$.

Remark: Concepts can be defined easily and clearly by using the language of sets.

Axioms of Incidence:

(I 1) For any distinct points A, B, there exists a unique line ℓ_{AB} containing A, B.

$\forall$ distinct $A, B \in S$, $\exists ! \, \ell_{AB} \in \mathcal{L}$, $A, B \in \ell_{AB}$

(Remark: $\exists ! x, P(x) \equiv \exists x, P(x) \wedge \forall y \, (P(y) \Rightarrow y = x)$)

(I.2) Every line contains at least two points.

$\forall \ell \in \mathcal{L} \quad |\ell| \geq 2$

(I.3) There exist three noncollinear points

$\exists$ distinct $A, B, C \in S$, $\forall \ell \in \mathcal{L}$, $\neg (A \in \ell \wedge B \in \ell \wedge C \in \ell)$

Remark: (I.1) guarantees "sufficient" lines;

(I.2) excludes the possibility of degenerate line;

(I.3) guarantees "sufficient" points in a geometry.

Definition 3.1.2

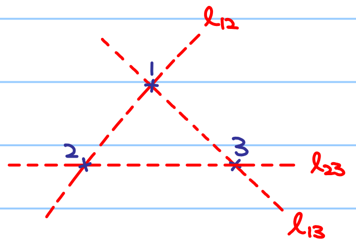
$(S, \mathcal{L})$ that satisfies the axioms (I.1)-(I.3) is called an incident geometry

Example 3.1.1

$S = \{1, 2, 3\}$

$\mathcal{L} = \{ \ell_{ij} = \{i, j\} : 1 \le i < j \le 3 \}$

$(S, \mathcal{L})$ is an incidence geometry.



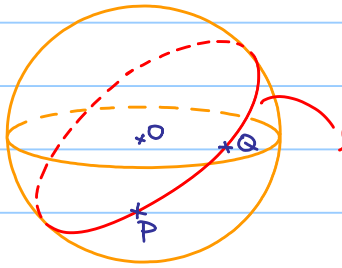
Caution : This picture only shows the idea, points in this geometry are not necessarily regarded as points on a plane and lines in this geometry is not lines on plane in usual sense.

Example 3.1.2

$S = S^2$ = set of points of a sphere

For each $\ell \in \mathcal{L}$, ℓ is a great circle

$(S, \mathcal{L})$ is not an incidence geometry. (Why?)



ℓ = great circle

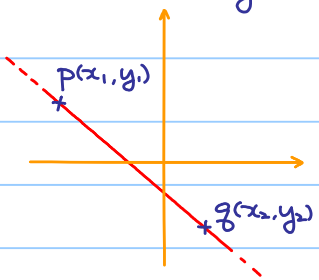
= plane passing through $O, P, Q \cap S^2$

Example 3.1.3

$S = F^2$, where $F = \mathbb{R}$.

$\mathcal{L} = \{$ equivalent equations in from of $ax + by + c = 0$: $a, b, c \in F$, not all of them are zero $\}$

$(S, \mathcal{L})$ is an incidence geometry. (Why?)



$\ell : (y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ if $x_1 \neq x_2$

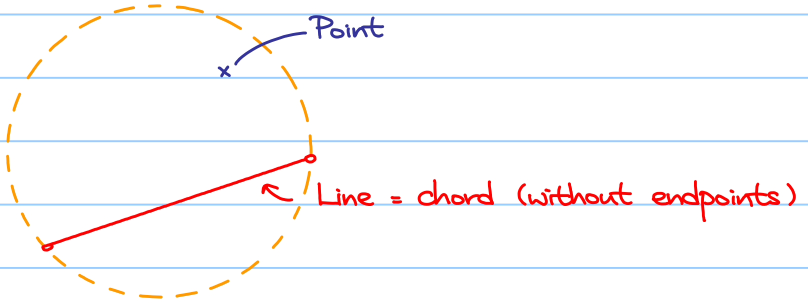
$x = x_1 = x_2$ if $x_1 = x_2$

How about changing S to be F^2 where $F = \mathbb{Q}$ or $\mathbb{C}$?

Example 3.1.5 (Klein Disk)

$S = \mathbb{D}^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$

$\mathcal{L} = \{\ell = L \cap \mathbb{D}^2 : L \text{ is a straight line in } \mathbb{R}^2\}$



$(S, \mathcal{L})$ is an incidence geometry. (Why?)

Proposition 3.1.1

Two distinct lines can have most one point in common.

$\ell, m \in \mathcal{L} \Rightarrow |\ell \cap m| \leq 1$

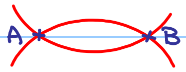
proof:

Suppose $|\ell \cap m| \geq 2$, then there exists $A, B \in S$ such that $A \neq B$ and $A, B \in \ell \cap m$.

I.1 $\Rightarrow$ $\ell = m$

Remark: The proof does NOT rely on any picture!

(Rather than saying that "two lines cannot enclose a space", see Euclid's proof of proposition 1.4)



Definition 3.1.3

Two distinct lines are parallel if they have no points in common. We also say that any line is parallel to itself.

If $\ell, m \in \mathcal{L}$ and $\ell \neq m$, then we define $\ell /\!/ m \Leftrightarrow \ell \cap m = \phi$. Also, we define $\ell /\!/ \ell$ $\forall \ell \in \mathcal{L}$.

Playfair's Axiom :

(P) For each point A and line ℓ, there exists at most one line m such that m passes through A and m is parallel to ℓ.

$\forall A \in S, \forall \ell \in \mathcal{L}, \exists$ at most one $m \in \mathcal{L}, A \in m \wedge m /\!/ \ell$

Think: How about $A \in \ell$?

If m is a line such that $A \in m$ and $m /\!/ \ell$, then $m \cap \ell \neq \phi$ which forces $m = \ell$, so the statement is satisfied automatically.

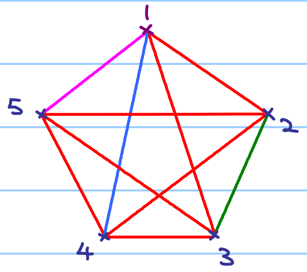
Example 3.1.5

$S = \{1, 2, 3, 4, 5\}$

$\mathcal{L} = \{ \ell_{ij} = \{i, j\} : 1 \leq i < j \leq 5 \}$

$(S, \mathcal{L})$ is an incidence geometry that does not satisfy (P).

Note : Consider the point 1 and the line ℓ_{23}, then $1 \in \ell_{14}, \ell_{15}$, $\ell_{23} /\!/ \ell_{14}$ and $\ell_{23} /\!/ \ell_{15}$



Exercise 3.1.1

Show that geometries in example 3.1.1 and 3.1.3 satisfies (P).

However, Klein disk in example 3.1.4 does not satisfy (P).

Exercise 3.1.2

If $(S, \mathcal{L})$ is an incidence geometry, does parallelism give an equivalence relation on $\mathcal{L}$?

Hint : consider example 3.1.5

Definition 3.1.4

Suppose that $(S_1, \mathcal{L}_1)$ and $(S_2, \mathcal{L}_2)$ are two geometries.

If there exists a bijective function $f : S_1 \to S_2$ and f induces a bijective function $F : \mathcal{L}_1 \to \mathcal{L}_2$ defined by $F(\ell) = \{ f(P) \in S_2 : P \in \ell \}$, then $(S_1, \mathcal{L}_1)$ and $(S_2, \mathcal{L}_2)$ are said to be isomorphic and f is said to be an isomorphism.

In particular, if $(S_1, \mathcal{L}_1) = (S_2, \mathcal{L}_2)$, f is said to be an automorphism.

Remark: Every geometry is isomorphic to itself by considering the identity function.

Example 3.1.6

$S_1 = \{ 1, 2, 3 \}$ $\qquad$ $S_2 = \{ A, B, C \}$

$\mathcal{L}_1 = \{ \ell_{ij} = \{ i, j \} : 1 \leq i < j \leq 3 \}$ $\qquad$ $\mathcal{L}_2 = \{ \ell_{AB} = \{A, B\}, \ell_{BC} = \{B, C\}, \ell_{AC} = \{A, C\} \}$

Let $f : S_1 \to S_2$ defined by $f(1) = A$, $f(2) = B$, $f(3) = C$.

Then f induces $F : \mathcal{L}_1 \to \mathcal{L}_2$ with $F(\ell_{12}) = \ell_{AB}$, $F(\ell_{23}) = \ell_{BC}$, $F(\ell_{13}) = \ell_{AC}$

This shows $(S_1, \mathcal{L}_1)$ and $(S_2, \mathcal{L}_2)$ are isomorphic.

Question: Any other isomorphisms?

Remark: When we say that "$(S_1, \mathcal{L}_1)$ and $(S_2, \mathcal{L}_2)$ are isomorphic", roughly speaking, they are having the same structure and f, F give a renaming of points and lines.

Example 3.1.7

$S = \mathbb{R}^2$

$\mathcal{L} = \{$ equivalent equations in from of $ax + by + c = 0$: $a, b, c \in \mathbb{R}$, not all of them are zero $\}$

Translations, rotations, reflections give automorphisms of $(S, \mathcal{L})$.

Discussion :

1) Independence of Axioms

We would like to show that each of the above axiom cannot be deduced from the others.

How to prove?

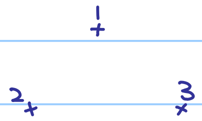
Easy! For each axiom, construct a model which satisfies all the other axioms.

Proposition 3.1.2

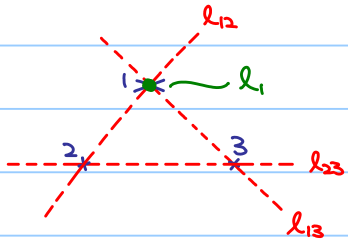
The axioms (I.1), (I.2), (I.3) and (P) are independent of each other

proof:

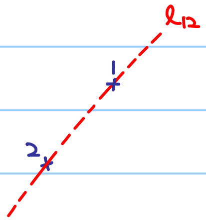
		satisfying
$S_1 = \{1,2,3\}$	$\mathcal{L}_1 = \phi$	(I.2), (I.3), (P)
$S_2 = \{1,2,3\}$	$\mathcal{L}_2 = \{\{1,2\},\{2,3\},\{1,3\},\{1\}\}$	(I.1), (I.3), (P)
$S_3 = \{1,2\}$	$\mathcal{L}_3 = \{\{1,2\}\}$	(I.1), (I.2), (P)
$S_4 = \{1,2,3,4,5\}$	$\mathcal{L}_4 = \{\, l_{ij} = \{i,j\} : 1 \leq i < j \leq 5 \}$	(I 1), (I.2), (I.3)



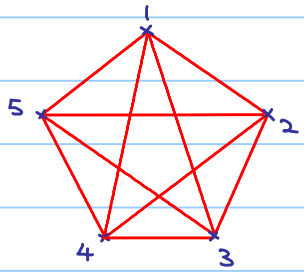
$(S_1, \mathcal{L}_1)$



$(S_2, \mathcal{L}_2)$



$(S_3, \mathcal{L}_3)$



$(S_4, \mathcal{L}_4)$

2) Uniqueness of model

In exercise 3.1.2, we show that both geometries in example 3.1.1 and 3.1.3 satisfy (I 1), (I.2), (I.3) and (P). If we further impose axioms, will it force the geometry obtained to be unique? In particular, what should we impose so that the unique geometry obtained is the Euclidean geometry?

3.2 Axioms of Betweenness

Here, we presuppose axioms (I.1), (I.2) and (I.3) of incidence geometry.

The geometrical notions of betweenness, separation, sidedness and order will all based on the following relation

Definition 3.2.1

Let $\mathbb{B} \subseteq \{(A,B,C) \in S^3 : A,B,C \text{ are distinct}\}$.

If $(A,B,C) \in \mathbb{B}$, we say that B is a point lying between A and C (but NOT C and A at this moment).

Then, we impose axioms on $\mathbb{B}$ such that it behaves as we expect

Axioms of Betweenness :

(B.1) If B is between A and C (written as $A*B*C$), then A, B, C lie on the same line and also $C*B*A$.

$(A,B,C) \in \mathbb{B} \Rightarrow \exists\, \ell \in \mathcal{L},\ A,B,C \in \ell \wedge (C,B,A) \in \mathbb{B}$

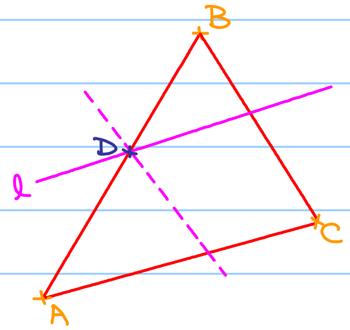
(B.2) For any two distinct points A, B, there exists a point C such that $A*B*C$

$\forall$ distinct $A,B \in S$, $\exists\, C \in S$, $(A,B,C) \in \mathbb{B}$.

(B.3) Given three distinct points on a line, one and only one of them is between the other two.*

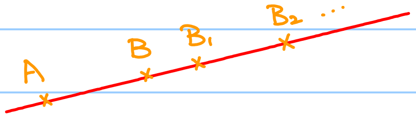
$$\begin{array}{lll} * \text{ Followed from (B.1)} & (C,B,A) \in \mathbb{B} & (A,C,B),\ (B,A,C) \notin \mathbb{B} \\ & \Uparrow \text{B.1} & \Uparrow \text{B.1} \\ \text{Given } A,B,C \in \ell \text{ for some } \ell \in \mathcal{L}. & (A,B,C) \in \mathbb{B} \Rightarrow & (B,C,A),\ (C,A,B) \notin \mathbb{B} \end{array}$$

(B.4) Let A, B, C be three noncollinear points, and let ℓ be a line not containing any A, B, C. If ℓ contains a point D lying between A and B, then it must also contains either a point lying between A and C or a point lying between B and C, but not both.



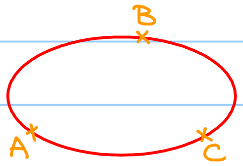
Remark : (B.1) says betweenness is a relation only defined for three distinct points on a line and symmetry of betweenness.

(B.2) guarantees extension of a line.

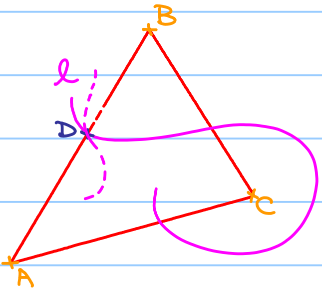


⇒ Exclude all finite geometries !

(B 3) avoids "looped line"



(B 4) avoids "curved line" ℓ



Definition 3.2.2

If A and B are distinct points, we define the line segment AB to be the set consisting of the points A, B and all points lying between A and B.

$(AB \overset{def}{=} \{C \in S . (A,C,B) \in \mathcal{B}\} \cup \{A,B\})$

Remark :

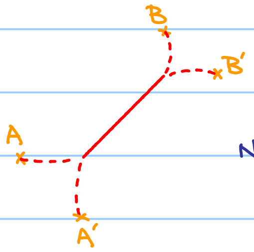
1) AB = BA by (B.1)

2) A line segment is uniquely determined by two endpoints.

Exercise 3.2.1

Given a line segment AB, show that there do not exist points C, D ∈ AB such that $C * A * D$.

(There are only two points of AB satisfying the above condition, namely A and B.)



NOT allowed !

3) (B.4) can be expressed as

$\forall$ distinct $A,B,C \in S$, $\forall \ell \in \mathcal{L}$ with $A,B,C \notin \ell$

$(\ell \cap AB \neq \phi) \rightarrow$ either $\ell \cap AC \neq \phi$ or $\ell \cap BC \neq \phi$

Exercise 3.2 2

Write down the contrapositive of (B.4)

Answer: $\forall$ distinct $A, B, C \in S$, $\forall \ell \in \mathcal{L}$ with $A, B, C \notin \ell$

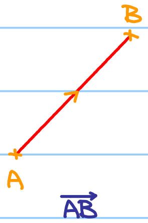
$$(\ell \cap AC, \ell \cap BC = \phi) \vee (\ell \cap AC, \ell \cap BC \neq \phi) \rightarrow (\ell \cap AB = \phi)$$

Note that a line segment is uniquely determined by two endpoints.

Let $\widetilde{\mathbb{P}}$ be the set of all oriented line segments in S, then

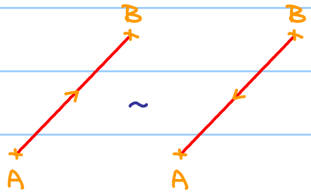
$\widetilde{\mathbb{P}} = \{(A,B) \in S^2 : A \neq B\}$. (To be precise, there exists a bijection between them.)

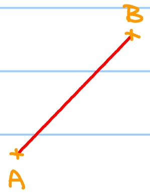
We may denote $(A,B) \in \widetilde{\mathbb{P}}$ by $\overrightarrow{AB}$.



Furthermore, by considering the equivalence relation on $\widetilde{\mathbb{P}}$ defined by $(A,B) \sim (B,A)$.

If $\mathbb{P}$ is the set of all line segments in S, then $\mathbb{P} = \widetilde{\mathbb{P}}/\sim$.





$\overrightarrow{AB}$ $\overrightarrow{BA}$ $AB = BA = [\overrightarrow{AB}] = [\overrightarrow{BA}]$

Remark: We need the notion of $\widetilde{\mathbb{P}}$ when we introduce addition of line segments.

Definition 3.2 3

Let A, B and C be three noncollinear points. A triangle is defined to be $AB \cup BC \cup CA$

Proposition 3 2 1 (Plane separation)

Let ℓ be a line. Then the set of points not lying on ℓ can be divided into two nonempty subsets S_1, S_2 with the following properties:

Then $\exists\ S_1, S_2 \subseteq S$ s.t. $S_1, S_2 \neq \phi$ and $S \setminus \ell = S_1 \amalg S_2$ with the following properties.

(i) Two points A, B not on ℓ belong to the same set (S_1 or S_2) if and only if the segment AB does not intersect ℓ.

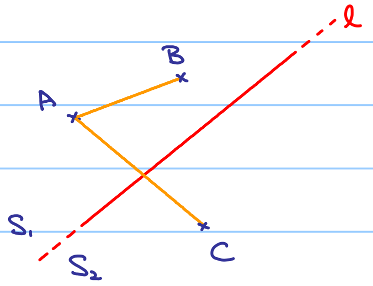
$\forall$ distinct $A, B \in S \setminus \ell$, $A, B \in S_1 \vee A, B \in S_2 \iff AB \cap \ell = \phi$

(ii) Two points A, C not on ℓ belong to the opposite sets (one in S_1, the other in S_2) if and only if the segment AC intersects ℓ at a point.

$\forall$ distinct $A, C \in S \setminus \ell$, $(A \in S_1, C \in S_2) \vee (A \in S_2, C \in S_1) \iff AC \cap \ell \neq \phi$

(Note: if $AC \cap \ell \neq \phi$, then since $AC \cap \ell \subseteq \ell_{AC} \cap \ell$ and $|\ell_{AC} \cap \ell| \leq 1$, $AC \cap \ell = \{*\}$)

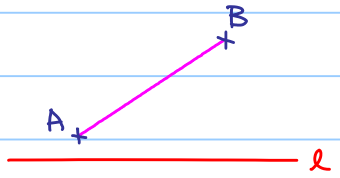
From the above, S_1, S_2 are said to be two sides of ℓ. Furthermore, if both A and B belong to the same set S_i, $i = 1$ or 2, then they are said to be on the same side of ℓ, otherwise they are said to be on the opposite side of ℓ.



proof:

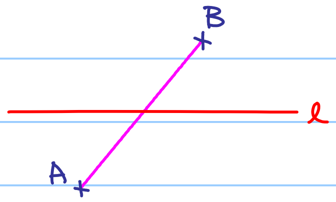
Idea: Define a relation $\sim$ on $S \setminus \ell$:

Let $A, B \in S \setminus \ell$ $\quad A \sim B \iff \begin{cases} AB \cap \ell = \phi & \text{if } A \neq B \\ A \sim A & \text{if } A = B \end{cases}$



same side of ℓ

Prove that $\sim$ is an equivalence relation and there are only two equivalence classes.



opposite side of ℓ

$A \sim A$ (By definition) ✓

$A \sim B \Rightarrow B \sim A$ ($\because AB = BA$) ✓

$A \sim B$ and $B \sim C \Rightarrow A \sim C$?

$\Big\}$ $\sim$ is an equivalence relation

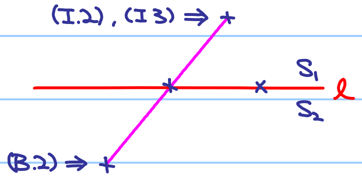
$A \nsim B$ and $B \nsim C \Rightarrow A \sim C$? $\}$ Only two equivalence classes

Exercise 3.2.3

Prove by (I.2) and (I.3) that for all line ℓ, there exists a point which does not lie on ℓ

By the above exercise, $S_1 \neq \phi$

(B.2) guarantees $S_2 \neq \phi$.



Case 1 : A, B, C are noncollinear

Contrapositive of (B.4) : $(\ell \cap AC, \ell \cap BC = \phi) \vee (\ell \cap AC, \ell \cap BC \neq \phi) \Rightarrow (\ell \cap AB = \phi)$

$A \sim B$ and $B \sim C \Rightarrow A \sim C$, $A \nsim B$ and $B \nsim C \Rightarrow A \sim C$

Case 2 : A, B, C are collinear

Idea : Show that there exists E such that

A, E, C are noncollinear, $A \sim E$ and $E \sim C$ —— (a) ($A \nsim E$ and $E \nsim C$ —— (b)),

case 1 $\Rightarrow A \sim C$.

proof of (a) and (b) :

see prop. 7.1 in [2].

Proposition 3.2.2 (Line separation)

Let A be a point on a line ℓ. Then the set of points of ℓ not equal to A can be divided into nonempty subsets r_1, r_2, the two sides of A on ℓ, such that

(i) B, C are on the same side if and only if A is not lying on the segment BC.

(ii) B, C are on the opposite side if and only if A is lying on the segment BC.

proof:

From (I.2), (I.3), there exists $D \in S$ such that $D \notin \ell$.

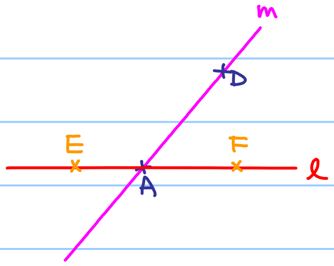
By (I.1), A and D determines a line m.

By proposition 3.2.1, $S \setminus m = S_1 \amalg S_2$ for some nonempty sets S_1, S_2. Let $r_i = S_i \cap \ell$, $i = 1, 2$.

(I.2) $\Rightarrow \exists\, E \in \ell$ and $E \neq A$, (B.2) $\Rightarrow \exists\, F \in \ell$ s.t. $E * A * F$.

$\therefore r_1, r_2 \neq \phi$

Remark: It defines an equivalence relation on $\ell \setminus \{A\}$.

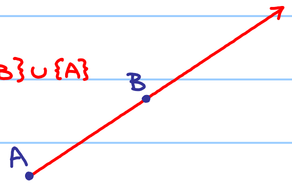


Definition 3.2.4

Given two distinct points A, B, the ray r_{AB} is the set consisting of A, plus all points on the line ℓ_{AB} that are on the same side of A as B

If ~ is the equivalence relation by the above remark, then $r_{AB} = \{C \in AB : C \sim B\} \cup \{A\}$

The point A is the origin, or vertex, of the ray

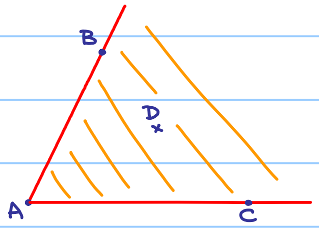


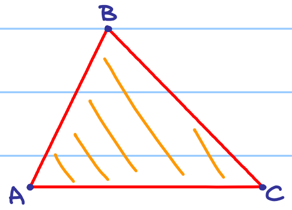
Definition 3.2.5

An angle is the union of two rays r_{AB} and r_{AC} originating at the same point, its vertex, and not lying on the same line. We denote the angle by $\angle BAC$ or $\angle CAB$.

Furthermore, the interior of an angle $\angle BAC$ consists of all points D such that D and C are on the same side of the line AB, and D and B are on the same side of the line AC.

If ABC is a triangle, the interior of the triangle ABC is the set of points that are simultaneously in the interiors of the three angles $\angle BAC$, $\angle ABC$, $\angle ACB$



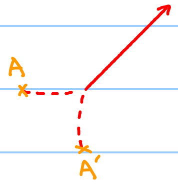


Remark: "Zero angle" and "straight angle" are not included in this definition.

Exercise 3.2.4

Show that the vertex of a ray (or an angle) is uniquely determined by the ray (or angle).

(Rephrase: there is only one vertex of a ray (or an angle).)



NOT allowed !

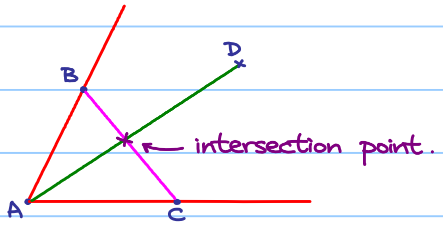
Proposition 3.2.3 (Crossbar theorem)

Let $\angle BAC$ be an angle, and let D be a point in the interior. Then the ray r_{AD} must meet the line segment BC.

proof:

see prop. 7.3 in [2].

Think: How about the uniqueness?



Example 3.2.1

Let $S=\mathbb{R}$ and $\mathcal{L}=\{\ell=\mathbb{R}\}$.

a) Show that it is not an incidence geometry.

b) Define the notion of betweenness in "usual" sense on $\mathbb{R}$: For three distinct real numbers a,b,c,

$a*b*c$ if $a<b<c$ or $c<b<a$.

Show that it satisfies (B.1) – (B.3)

Example 3.2.2

Let $S=\mathbb{R}^2$, $A=(a_1,a_2)$, $B=(b_1,b_2)$ and $C=(c_1,c_2)$ be three distinct points in $\mathbb{R}^2$

Define $A*B*C$ if A,B,C are on the same line such that either $a_1*b_1*c_1$ or $a_2*b_2*c_2$ or both

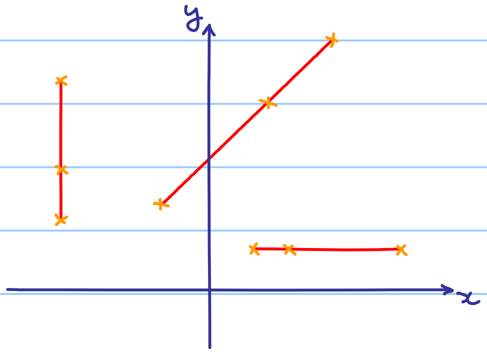
(Note: $*$ is defined on $\mathbb{R}^2$, but $*$ is defined on $\mathbb{R}$ in example 3.2.1)

Show that it satisfies (B.1) – (B.4)

Is it still true if we change S to be $\mathbb{Q}^2$?

Answer: Yes!

Think: What happens in each case?



Example 3.2.3

Define the betweenness of points of Klein disk by regarding them as points in $\mathbb{R}^2$.

Show that it satisfies (B.1) – (B.4)

3.3 Axioms of Congruence for Line Segments

Next, we postulate an undefined notion of congruence, which is a relation between two line segments AB and CD, denoted by $AB \cong CD$.

Definition 3.3.1

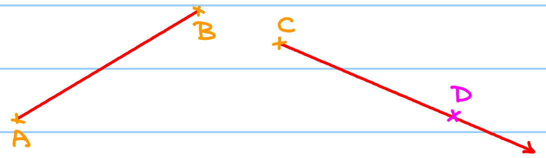
Let $\mathcal{C} \subseteq \mathcal{P} \times \mathcal{P}$. If $(s_1, s_2) \in \mathcal{C}$, then s_1 is said to be congruent to s_2, and we denote it by $s_1 \cong s_2$.
If $s_1 = AB$, $s_2 = CD$, we write $(AB, CD) \in \mathcal{C}$ or $AB \cong CD$.

Remark: We usually write $AB = CD$ instead of $AB \cong CD$. However, here we regard them as subsets of S and they are different as sets, that is why we put "$\cong$" here.

Then, we impose axioms on $\mathcal{C}$ such that it behaves as we expect.

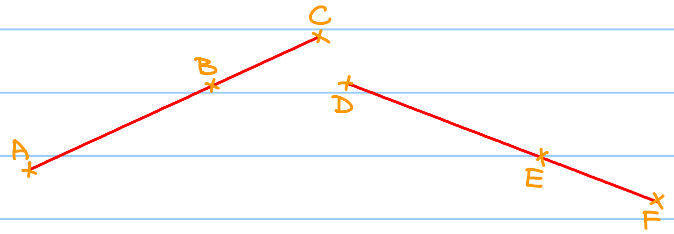
Axioms of Congruence for Line Segments:

(C.1) Given a line segment AB, and given a ray r originating at a point C, there exists a unique point D on the ray r such that $AB \cong CD$.



(C.2) If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Every line segment is congruent to itself.

(C.3) Given three distinct points A, B, C on a line satisfying $A * B * C$, and three further points D, E, F on a line satisfying $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.



Roughly: $AB \cong DE$, $BC \cong EF \Rightarrow AB \text{``}+\text{''} BC = AC \cong DF = DE \text{``}+\text{''} EF$ which replaces Euclid's second common notion.

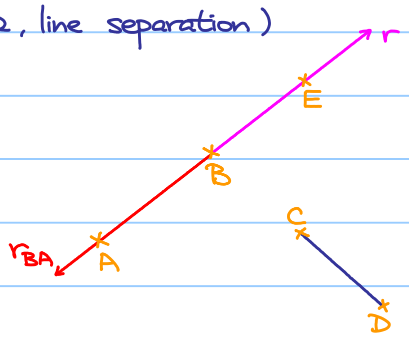
Remark: (C 1) acts as "transporter of segments".

(C.2) says that congruence is an equivalence relation on $\mathcal{P}$. (See also exercise 2.2.2)

Addition :

Consider the ray r which is opposite to r_{BA} (proposition 3.2.2, line separation)

By (C.1), there exists unique $E \in r$ such that $BE \cong CD$.



Definition 3.3.2

Addition on $\widetilde{P}$, $+ : \widetilde{P} \times \widetilde{P} \longrightarrow \widetilde{P}$, is defined by

$\overrightarrow{AB} + \overrightarrow{CD} := \overrightarrow{AE}$

Recall :

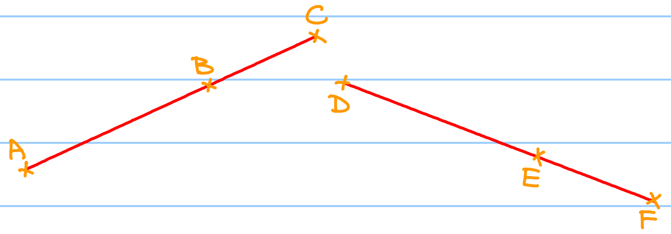
Let $\widetilde{P}$ be the set of all oriented line segments in S.

By considering the equivalence relation on $\widetilde{P}$ defined by $\overrightarrow{AB} \sim \overrightarrow{BA}$,

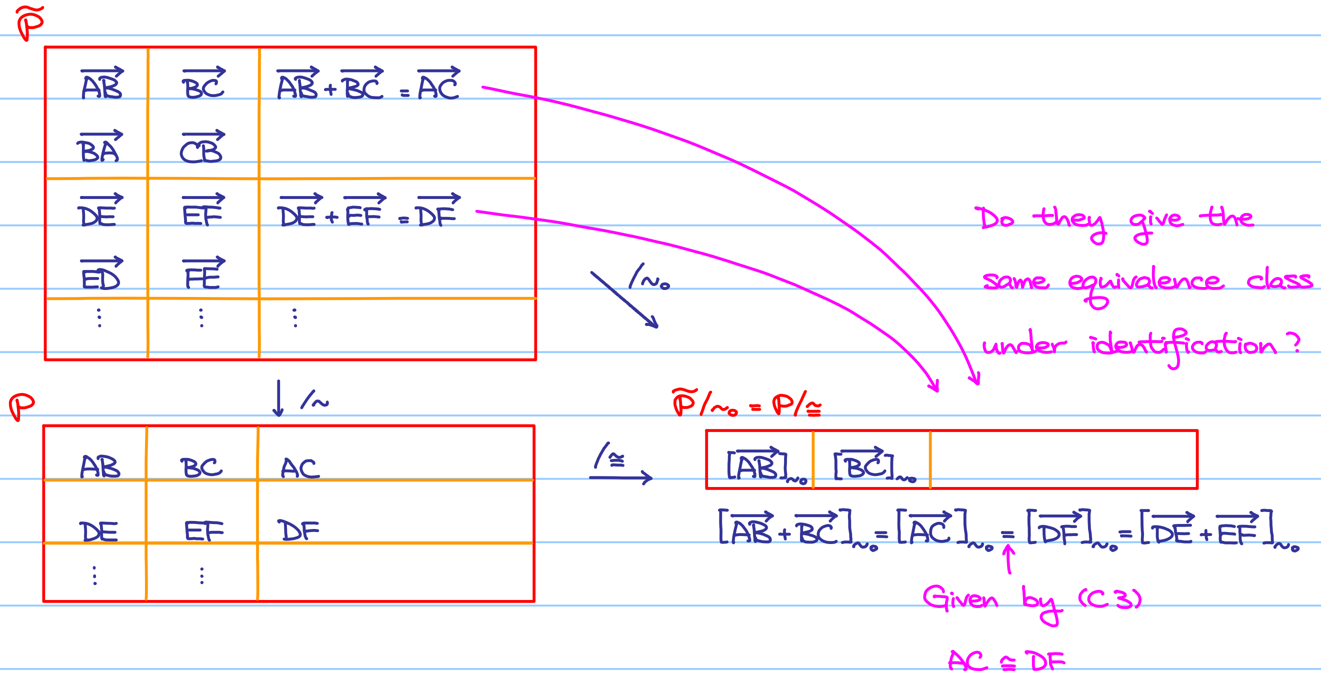
then $P = \widetilde{P}/\sim$. where P is the set of all line segments in S.

Furthermore, $\cong$ is an equivalence relation on P by (C.2).

It gives an equivalence relation $\sim_0$ on $\widetilde{P}$ where $\widetilde{P}/\sim_0 = (\widetilde{P}/\sim)/\cong = P/\cong$.



$AB \cong DE$ and $BC \cong EF$



$\widetilde{P}$

$\overrightarrow{AB}$	$\overrightarrow{BC}$	$\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$
$\overrightarrow{BA}$	$\overrightarrow{CB}$	
$\overrightarrow{DE}$	$\overrightarrow{EF}$	$\overrightarrow{DE}+\overrightarrow{EF} = \overrightarrow{DF}$
$\overrightarrow{ED}$	$\overrightarrow{FE}$	
⋮	⋮	⋮

$/\sim_0$

$/\sim$

P

AB	BC	AC
DE	EF	DF
⋮	⋮	

$/\cong$

$\widetilde{P}/\sim_0 = P/\cong$

$[\overrightarrow{AB}]_{\sim_0}$	$[\overrightarrow{BC}]_{\sim_0}$	

Do they give the same equivalence class under identification ?

$[\overrightarrow{AB}+\overrightarrow{BC}]_{\sim_0} = [\overrightarrow{AC}]_{\sim_0} = [\overrightarrow{DF}]_{\sim_0} = [\overrightarrow{DE}+\overrightarrow{EF}]_{\sim_0}$

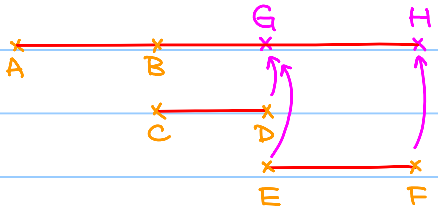
Given by (C3)

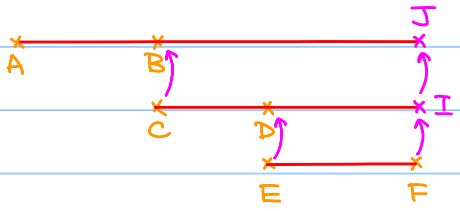
$AC \cong DF$

Addition can be defined on $\widetilde{\mathbb{P}}$, but it is "bad" in the sense that $\overrightarrow{AB} + \overrightarrow{CD} \neq \overrightarrow{CD} + \overrightarrow{AB}$ (not commutative). What we only have is associativity.

Exercise 3.3.1

Show that the addition in definition 3.3.2 is associative, i.e. $(\overrightarrow{AB} + \overrightarrow{CD}) + \overrightarrow{EF} = \overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF})$.





$$(\overrightarrow{AB} + \overrightarrow{CD}) + \overrightarrow{EF} \overset{?}{=} \overrightarrow{AB} + (\overrightarrow{CD} + \overrightarrow{EF})$$

Why $H = J$?

We would like to show that addition defined on $\widetilde{\mathbb{P}}$ descends to $\widetilde{\mathbb{P}}/\sim_0 = \mathbb{P}/\cong$.
Also addition on $\mathbb{P}/\cong$ behaves "nicely" as expected.

Proposition 3.3.1

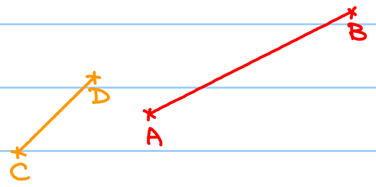
Addition defined on $\widetilde{\mathbb{P}}$ descends to $\widetilde{\mathbb{P}}/\sim_0 = \mathbb{P}/\cong$.

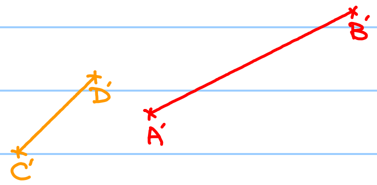
proof:

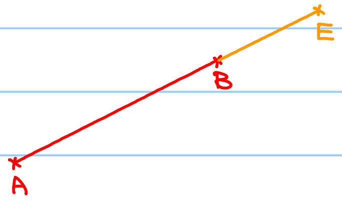
Recall theorem 2.4.1 and discussion in section 2.4, what we have to show is :

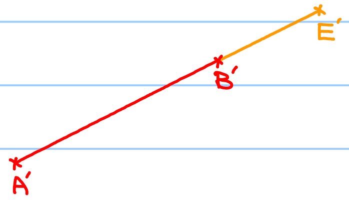
$\overrightarrow{AB} \sim_0 \overrightarrow{A'B'}$, $\overrightarrow{CD} \sim_0 \overrightarrow{C'D'} \Rightarrow \overrightarrow{AB} + \overrightarrow{CD} \sim_0 \overrightarrow{A'B'} + \overrightarrow{C'D'}$.

(see also prop. 8.2 in [2])









$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{AB} + \overrightarrow{CD}$ (by definition) $\quad\overrightarrow{A'E'} = \overrightarrow{A'B'} + \overrightarrow{B'E'} = \overrightarrow{A'B'} + \overrightarrow{C'D'}$ (by definition)

$BE \cong CD \cong C'D' \cong B'E'$ and $AB \cong A'B' \xrightarrow{(C.3)} AE \cong A'E'$

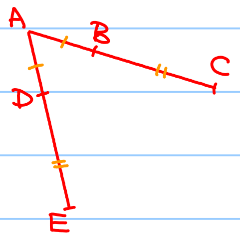
i.e. $\overrightarrow{AB} + \overrightarrow{CD} \sim_0 \overrightarrow{A'B'} + \overrightarrow{C'D'}$.

Furthermore, Euclid's second common notion

> If equals are added to equals, then the wholes are equal.

is rephrased as the following:

For example, in Euclid's Elements,



$$AB = AD$$

$$BC = DE$$

$$\text{so} \quad AB+BC = AD+DE \qquad (\text{C.N.2})$$

$$AC = AE$$

With proposition 3.3.1, in our language,

$$AB \cong AD \qquad (\text{i.e. } [AB]_{\cong} = [AD]_{\cong} \text{ in } \mathbb{P}/{\cong})$$

$$BC \cong DE$$

$$\text{so} \quad [AB]_{\cong} + [BC]_{\cong} = [AD]_{\cong} + [DE]_{\cong} \qquad (\text{here } + \text{ is the addition defined on } \mathbb{P}/{\cong})$$

$$[\overrightarrow{AB}]_{\sim_o} + [\overrightarrow{BC}]_{\sim_o} = [\overrightarrow{AD}]_{\sim_o} + [\overrightarrow{DE}]_{\sim_o} \qquad (\overrightarrow{AB} \text{ is a representative of } [AB]_{\cong} = [\overrightarrow{AB}]_{\sim_o})$$

$$[\overrightarrow{AB} + \overrightarrow{BC}]_{\sim_o} = [\overrightarrow{AD} + \overrightarrow{DE}]_{\sim_o} \qquad (\text{here } + \text{ is the addition defined on } \widetilde{\mathbb{P}})$$

$$[\overrightarrow{AC}]_{\sim_o} = [\overrightarrow{AE}]_{\sim_o}$$

$$AC \cong AE$$

Also addition on $\widetilde{\mathbb{P}}/{\cong}$ behaves "nicely" in the following sense:

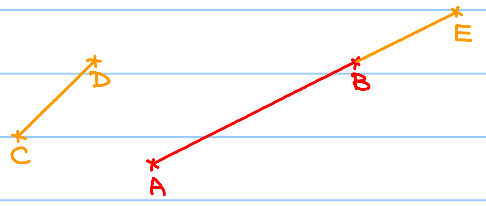
Proposition 3.3.2

Addition defined on $\mathbb{P}/{\cong}$ is associative and commutative.

proof:

Addition defined on $\mathbb{P}/{\cong}$ is associative which follows from the fact that addition defined on $\mathbb{P}$ is associative.

$$
\begin{aligned}
[AB]_{\cong} + [CD]_{\cong} &= [\overrightarrow{AB}]_{\sim_0} + [\overrightarrow{CD}]_{\sim_0} \\
&= [\overrightarrow{AB} + \overrightarrow{CD}]_{\sim_0} \\
&= [\overrightarrow{AE}]_{\sim_0} \qquad \text{where } CD \cong BE \\
&= [\overrightarrow{EA}]_{\sim_0} \\
&= [\overrightarrow{EB} + \overrightarrow{BA}]_{\sim_0} \\
&= [\overrightarrow{EB}]_{\sim_0} + [\overrightarrow{BA}]_{\sim_0} \\
&= [\overrightarrow{CD}]_{\sim_0} + [\overrightarrow{AB}]_{\sim_0} \qquad (CD \cong BE \Rightarrow \overrightarrow{CD} \sim_0 \overrightarrow{EB}) \\
&= [CD]_{\cong} + [AB]_{\cong}
\end{aligned}
$$



. . Addition defined on $\mathbb{P}/\cong$ is commutative

Comparison :

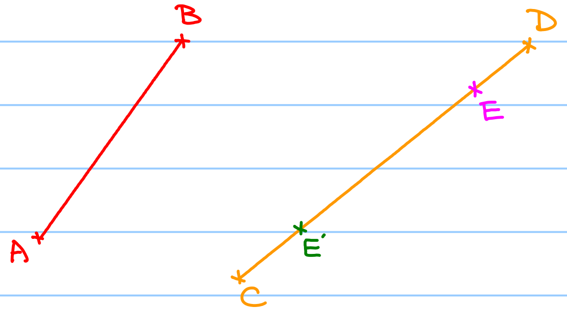
Definition 3.3.3

Let $AB, CD \in \mathbb{P}$.

AB is said to be less than CD if there exists E such that $C*E*D$ and $AB \cong CE$.

We denote it by $AB < CD$.

(Be careful, this definition depends on the choice of orientation of CD, so it is not well-defined unless we can show AB is less than CD if and only if AB is less than DC.)



$C*E*D$ and $AB \cong CE$.

Exercise 3.3.2

Prove that there exists E such that $C*E*D$ and $AB \cong CE$ if and only if there exists E′ such that $D*E'*C$ and $AB \cong DE'$.

Proposition 3.3.2

$<$ defines a relation on $\mathbb{P}$ that satisfies

1) If $AB < CD$ and $CD < EF$, then $AB < EF$

2) Given any $AB, CD \in \mathbb{P}$, one and only one of the following holds :

$AB < CD$, $AB \cong CD$, $AB > CD$

proof :

see prop. 8.4(b) in [2]

Definition 3.3.4

$<$ is a relation on $\mathcal{P}/\cong$ defined by $[AB]_{\cong} < [CD]_{\cong}$ if and only if $AB < CD$.

(but is it well-defined ? Will it happen that $AB < CD$, but there exist another representatives $A'B' \in [AB]_{\cong}$ and $C'D' \in [CD]_{\cong}$ such that $A'B' < C'D'$ is not true ?)

Proposition 3.3.3

Given $AB \cong A'B'$ and $CD \cong C'D'$ $AB < CD$ if and only if $A'B' < C'D'$.

proof :

see prop. 8.4(a) in [2]

Furthermore, we define $\leq$ on $\mathcal{P}/\cong$ by $[AB]_{\cong} \leq [CD]_{\cong}$ if and only if $[AB]_{\cong} < [CD]_{\cong}$ or $[AB]_{\cong} = [CD]_{\cong}$

To summarize $<$ and $\leq$ as what we expected :

Exercise 3.3.3 / Proposition 3.3.4

$\leq$ defines a total order relation and $<$ defines a strict total order relation on $\mathcal{P}/\cong$

proof :

1) (antisymmetric) Suppose that $[AB]_{\cong} \leq [CD]_{\cong}$ and $[AB]_{\cong} \geq [CD]_{\cong}$

$$[AB]_{\cong} \leq [CD]_{\cong} \Rightarrow [AB]_{\cong} < [CD]_{\cong} \text{ or } [AB]_{\cong} = [CD]_{\cong} \Rightarrow AB < CD \text{ or } AB \cong CD$$

and $$[AB]_{\cong} \geq [CD]_{\cong} \Rightarrow [AB]_{\cong} \geq [CD]_{\cong} \text{ or } [AB]_{\cong} = [CD]_{\cong} \Rightarrow AB > CD \text{ or } AB \cong CD$$

By proposition 3.3.2 (2), it can only be the case $AB \cong CD$, i.e. $[AB]_{\cong} = [CD]_{\cong}$.

2) (transitive) Suppose that $[AB]_{\cong} \leq [CD]_{\cong}$ and $[CD]_{\cong} \leq [EF]_{\cong}$

$$\begin{array}{lllll} [AB]_{\cong} \leq [CD]_{\cong} & \Rightarrow & [AB]_{\cong} < [CD]_{\cong} & \text{or} & [AB]_{\cong} = [CD]_{\cong} \\ & & 1 \quad 2 & 3 & 4 \\ [CD]_{\cong} \leq [EF]_{\cong} & \Rightarrow & [CD]_{\cong} < [EF]_{\cong} & \text{or} & [CD]_{\cong} = [EF]_{\cong} \end{array}$$

case 1 : $AB < CD$ and $CD < EF$, prop. 3.3.2 (1) $\Rightarrow AB < EF \Rightarrow [AB]_{\cong} < [EF]_{\cong}$

case 2, case 3 : $[AB]_{\cong} < [CD]_{\cong}$

case 4 : $[AB]_{\cong} = [CD]_{\cong} = [EF]_{\cong}$

$\therefore$ $[AB]_{\cong} \leq [EF]_{\cong}$

3) (totality) Given any $[AB]_{\cong}, [CD]_{\cong} \in \mathcal{P}$,

choose representatives $AB \in [AB]_{\cong}$ and $CD \in [CD]_{\cong}$.

By proposition 3.3.2 (2), one and only one of $AB < CD$, $AB \cong CD$, $AB > CD$ is true

$$\begin{array}{ccc} AB < CD & AB \cong CD & AB > CD \\ \Updownarrow & \Updownarrow & \Updownarrow \\ [AB]_{\cong} < [CD]_{\cong} & [AB]_{\cong} = [CD]_{\cong} & [AB]_{\cong} > [CD]_{\cong} \end{array}$$

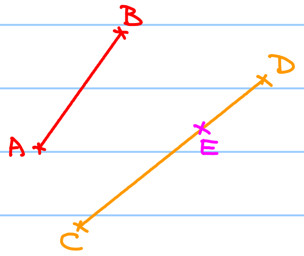
$\therefore$ $[AB]_{\cong} \leq [CD]_{\cong}$ or $[AB]_{\cong} \geq [CD]_{\cong}$

Subtraction :

Main trouble : If $AB \geqslant CD$, we do not know how to define $CD - AB$!

Recall : If $AB < CD$, then there exists E such that $C * E * D$ and $AB \cong CE$.

Definition 3.3.4



$- : \{(\overrightarrow{AB}, \overrightarrow{CD}) \in \widetilde{\mathcal{P}} \times \widetilde{\mathcal{P}} : AB < CD\} \longrightarrow \widetilde{\mathcal{P}}$ is defined by

$\overrightarrow{CD} - \overrightarrow{AB} := \overrightarrow{ED}$

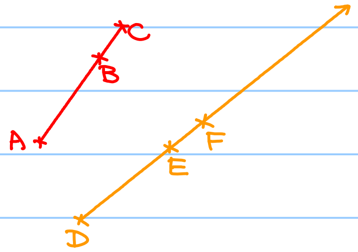
Proposition 3.3.5

Given three points A, B, C on a line such that $A * B * C$, and given points E, F on a ray originating from D, suppose that $AB \cong DE$ and $AC \cong DF$. Then $D * E * F$ and $BC \cong EF$.

proof :

see prop. 8.3 in [2]

(Compare to axiom (C 3))



Proposition 3.3.5 serves as a substitute of Euclid's third common notion :

> If equals are subtracted from equals, then the remainders are equal.

Proposition 3.3.6

$- : \{(\overrightarrow{AB}, \overrightarrow{CD}) \in \widetilde{\mathcal{P}} \times \widetilde{\mathcal{P}} : AB < CD\} \longrightarrow \widetilde{\mathcal{P}}$ descends to $\widetilde{\mathcal{P}}/\sim_0 = \mathcal{P}/_{\cong}$

$- : \{([\overrightarrow{AB}]_{\cong}, [\overrightarrow{CD}]_{\cong}) \in \mathcal{P}/_{\cong} \times \mathcal{P}/_{\cong} : [\overrightarrow{AB}]_{\cong} < [\overrightarrow{CD}]_{\cong} \} \longrightarrow \mathcal{P}/_{\cong}$

proof :

Similar to the proof of proposition 3.3.1

what we have to show is :

$\overrightarrow{AB} \sim_0 \overrightarrow{A'B'}, \overrightarrow{CD} \sim_0 \overrightarrow{C'D'} \Rightarrow \overrightarrow{CD} - \overrightarrow{AB} \sim_0 \overrightarrow{C'D'} - \overrightarrow{A'B'}$.

Apply proposition 3.3.5